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1 January 28, 2026

1.1 Periodic orbits of circle rotations

Let $X = [0, 1)$ and fix $\theta \in \mathbb{R}$. Define the map $T_\theta : X \rightarrow X$ by

$$T_\theta(x) = x + \theta \pmod{1} = \underbrace{x + \theta - [x + \theta]}_{\text{(other commonly used notations)}} = \{x + \theta\}. \tag{1}$$

Question/Problem 1.1. As we saw in class, every point in X is periodic if $\theta \in \mathbb{Q}$. What if $\theta \in \mathbb{R} \setminus \mathbb{Q}$? Are there periodic orbits? Recurrent orbits? Dense orbits? Try to prove your answers.

And something to think about, because eventually it will come up:

Question/Problem 1.2. What of equidistribution? True or false: For every sub-interval $J \subseteq [0, 1)$, and $x \in [0, 1)$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{1}_J(T_\theta^k x) = |J|, \tag{2}$$

where $|J|$ denotes the length of the interval J , and $\mathbb{1}_J$ is the indicator function

$$\mathbb{1}_J(x) = \begin{cases} 1 & \text{if } x \in J \\ 0 & \text{if } x \notin J. \end{cases}$$

Does (2) hold for some x and not others? Are there conditions on θ upon which the answer depends?

1.2 Periodic orbits of circle expanding maps

Let $m \geq 2$ be an integer. Let $X = [0, 1)$ and define the map $T_m : X \rightarrow X$ by

$$T_m(x) = mx \pmod{1} = mx - [mx] = \{mx\}. \tag{3}$$

For example, T_2 is the circle doubling map from class. As we saw, T_2 has a fixed point, and it also has “sinks,” and it has at least one periodic orbit:

$$\begin{array}{ll}
 0 \mapsto 0 \mapsto 0 \mapsto \dots & \text{(fixed point)} \\
 \frac{1}{2} \mapsto 0 \mapsto 0 \mapsto \dots & \text{(some sort of sink behavior)} \\
 \frac{1}{8} \mapsto \frac{1}{4} \mapsto \frac{1}{2} \mapsto 0 \mapsto 0 \mapsto \dots & \text{(another, just for kicks)} \\
 \frac{1}{3} \mapsto \frac{2}{3} \mapsto \frac{1}{3} \mapsto \frac{2}{3} \mapsto \dots & \text{(periodic orbit)} \\
 \frac{1}{6} \mapsto \frac{1}{3} \mapsto \frac{2}{3} \mapsto \frac{1}{3} \mapsto \frac{2}{3} \mapsto \dots & \text{(something else)}
 \end{array}$$

In contrast to what we saw with rotations, here it seems that orbits do not all look the same.

Question/Problem 1.3. Can you characterize the periodic points for T_m ? (Or specifically T_2 if you prefer.) If $x \in [0, 1)$ is periodic, does it follow that x is irrational? What other behaviors are possible for the orbits of T_m ? Are there recurrent orbits? Dense orbits? Equidistributed orbits? Fractal orbits? (Surprise!)

1.3 Circles, circles, circles, ...circles

Why do I keep calling $[0, 1)$ a circle, when we can all clearly see that it is not a circle but a line segment? Why are we okay with that? What’s really going on?

If you’ve taken topology you might be inclined to write $[0, 1]/\sim$ where \sim is an equivalence relation identifying $0 \sim 1$ as a single point. One would say that 0 has been glued to 1 , and that the resulting space is a circle.

If you’ve taken abstract algebra, you might recognize that $[0, 1)$ is a fundamental domain for the natural \mathbb{Z} -action on \mathbb{R} . Every element of the coset space \mathbb{R}/\mathbb{Z} has a unique representative in $[0, 1)$. Another way to express (1) and (3) would be as maps $\mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$ defined by

$$\begin{aligned}
 T_\theta(x + \mathbb{Z}) &= (x + \theta) + \mathbb{Z} \\
 T_m(x + \mathbb{Z}) &= mx + \mathbb{Z}.
 \end{aligned}$$

Sometimes you see that. The coset space \mathbb{R}/\mathbb{Z} , endowed with the “quotient topology” coming from \mathbb{R} , is a circle. Imagine the real line has been wound up into the shape of a circle, with all integers falling on the same point.

If you’ve taken complex analysis, you might prefer the circle as

$$\{z \in \mathbb{C} : |z| = 1\},$$

the set of points in the complex plane that are a distance 1 from $0 \in \mathbb{C}$. Taking this view, we would write

$$\begin{aligned}
 T_\theta(z) &= e^{2\pi i \theta} z \\
 T_m(z) &= z^m
 \end{aligned}$$

for the maps (1) and (3).

If you prefer your circles the traditional way, and you've taken linear algebra, then maybe your circle is defined by

$$x^2 + y^2 = 1, \tag{4}$$

by which one really means

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

the set of points in the plane satisfying (4). How to express (1) and (3)? The rotation is easy enough:

$$T_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

What about the doubling map?

You're free to use any of these points of view to answer the questions in §1.1 and §1.2. Sometimes, picking the right one view makes the whole problem way easier.

1.4 Some answers

1.4.1 Question/Problem 1.1

Several of you saw the relevance of the pigeonhole principle. Let $\varepsilon > 0$ and $x \in [0, 1)$. Cover $[0, 1)$ by finitely many intervals of length $< \varepsilon$. As soon as N exceeds the number of intervals, the pigeonhole principle guarantees that there exist $1 \leq k < \ell \leq N$ such that kx and ℓx occupy the same interval from the cover. In particular, $|\ell x - kx| < \varepsilon$. But $\ell x - kx = T_\theta^{\ell-k} x$. We find the the sub-orbit $\{T_\theta^{n(\ell-k)}\}_{n \geq 1}$ is ε -dense, meaning that every ε -neighborhood in $[0, 1)$ contains a point of it. Since $\varepsilon > 0$ was arbitrary, density follows.

Some of you presented a dual point of view that is illuminating. Assume that the T_θ -orbit of x is not dense. Then there must be some positive-length interval $I \subseteq [0, 1)$ that is not visited by the orbit. It follows that the sets

$$T_\theta^{-1}I, T_\theta^{-2}I, T_\theta^{-3}I, \dots$$

are also not visited. Now consider their union,

$$\bigcup_{k \geq 0} T_\theta^{-k}I.$$

What does it look like? The observation you made was that you run out of room! The union must be the whole interval!

1.4.2 Question/Problem 1.3

Great discussions on this one! Let $m \geq 2$ and consider T_m , the $\times m$ map of the circle. A periodic point x is characterized by the existence of some $k \geq 1$ such that

$$x = m^k x \pmod{1}.$$

It is easier to phrase this in terms of complex numbers, where the unit circle is $\{z \in \mathbb{C} : |z| = 1\}$. Now we look for z such that

$$z^{m^k} = z.$$

The periodic points are the $(m^k - 1)$ th roots of unity, solutions to $z^{m^k - 1} = 1$.

Here's a follow-up question some of you thought about, and others should! Can you tell me the periodic points of period k , *i.e.* the points that return on the k th iterate and no sooner?

2 February 2, 2026

2.1 Toral translations

Fix $\theta \in \mathbb{R}^2$ and let $T : \mathbb{T} \rightarrow \mathbb{T}$ be the toral translation $x \mapsto (x + \theta) \bmod 1$.

Question/Problem 2.1. Under what conditions on θ does this system have periodic orbits? What about dense orbits? Can you think of choices of θ such that the toral translation by θ has orbits that are neither dense nor periodic?

2.2 Cat map

Let $T : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be given by

$$T(x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x \bmod 1,$$

where “mod 1” means that whatever lands outside of the unit square gets identified back into it by shifting by elements of \mathbb{Z}^2 .

When I drew this in class, the parallelogram I sketched was inaccurate in various respects. Draw it with more care, do the cutting and pasting, convince yourself that the map as I described it really does make sense!

Question/Problem 2.2. Does the cat map have periodic orbits?

2.3 Symbolic systems

Let $X = \{0, 1\}^{\mathbb{N}}$, the collection of 0-1 sequences, and T the left shift. That is,

$$T(\omega_1 \omega_2 \omega_3 \dots) = \omega_2 \omega_3 \omega_4 \dots$$

Question/Problem 2.3. Does T have periodic orbits? Characterize them. What would it mean to say that two points in X are near each other? Can you think of any points in X whose T -orbits are dense with respect to some natural notion of “closeness”?

Some answers. Several of you had good ideas to address these questions. The first question is simple enough. An element of $\{0, 1\}^{\mathbb{N}}$ has a (pre)periodic orbit if it is a (pre)periodic sequence.

Regarding “closeness,” I saw from many of you the right idea. The sequences ω and ω' should be considered close if they agree in their first many entries. One can even quantify this. I saw from Ethan the suggestion

$$d(\omega, \omega') = 2^{-\nu},$$

where

$$\nu = \min\{n \geq 0 : \omega_n \neq \omega'_n\}.$$

Makes sense to me, and it seems to reflect distance in $[0, 1]$, if we are to interpret 0-1-sequences as binary expansions.

The idea can be pushed. A topology can be introduced on $\{0, 1\}^{\mathbb{N}}$ using as basis *cylinder sets*—subsets of the sequence space consisting of sequences with prescribed entries in finitely many positions.

Can you find dense orbits for this topology? □

2.4 Preview of measure-preservingness

Eventually, we will have a well-formed notion of “measure,” assigning to subsets $A \subseteq X$ a size $\mu(A)$, and a transformation $T : X \rightarrow X$ will be called measure-preserving if for every measurable $A \subseteq X$ one has $\mu(T^{-1}(A)) = \mu(A)$.

3 February 4, 2026

3.1 Outer measure on \mathbb{R}

We defined the outer measure of a set $A \subseteq X$ to be

$$\lambda^*(A) = \inf \left\{ \sum_{k=1}^{\infty} |I_k| : \text{each } I_k \text{ is a bounded interval and } A \subseteq \bigcup_k I_k \right\},$$

and stated a proposition claiming that outer measure satisfies the following properties:

1. It is equivalently defined using bounded *open* intervals.
2. It is equivalently defined using uniformly bounded intervals, that is, $|I_k| < \delta$ for some $\delta > 0$.
3. It is monotonic: If $A \subseteq B$ then $\lambda^*(A) \leq \lambda^*(B)$.
4. It is countably subadditive: For any $A_1, A_2, A_3, \dots \subseteq X$,

$$\lambda^* \left(\bigcup_{k=1}^{\infty} A_k \right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k).$$

We proved the first of these. Try the other ones!

All of this is presented as an extension of the concept of *length of an interval* to sets other than intervals. But...

Question/Problem 3.1. Is it true that

$$\lambda^*(I) = |I|$$

for every interval I ? I hope so!

3.2 Asides

There were a couple of asides that you might find interesting to read about.

Fractals! Cantor's dust, Sierpinski's gasket, Menger's sponge, Koch's snowflake, Mandelbrot's...set? One can spend an entire semester investigating the various properties of these fascinating sets.

The term "fractal" was coined by the eponymous Benoit B. Mandelbrot. Can you guess what the B stands for?

Banach–Tarski paradox Another fascinating thing to read about. You can chop a ball into finitely many pieces, and by rigid motions rearrange those pieces into two copies of the original ball. The pieces are "nonmeasurable" in the sense that I alluded to in class, but still, what an amazing result!

By the way, there's an equally amazing anagram of "Banach–Tarski." Can you find it?

4 February 9, 2026

4.1 Null sets

A set $N \subseteq \mathbb{R}$ is called a *null set* if $\lambda^*(N) = 0$. We showed that singletons are null sets. And, by sub-additivity, countable sets are null sets. We also saw that the Cantor set is a null set. I stated:

Theorem. *The Cantor set is a null set and is uncountable and contains no nontrivial intervals and is perfect.*

As set is called *perfect* if it is closed and has no isolated points.

Question/Problem 4.1. Prove that the Cantor set is perfect.

Question/Problem 4.2. Is it possible for a subset of \mathbb{R} to be uncountable and consist only of isolated points?

4.2 The tent map

The tent map is $T : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$T(x) = \begin{cases} 3x & \text{if } x \leq 1/2 \\ 3 - 3x & \text{if } x > 1/2. \end{cases}$$

Let $\Lambda = \{x \in \mathbb{R} : x \text{ has a bounded } T\text{-orbit}\}$. As we saw, it is the Cantor set.

5 February 11, 2026

5.1 Invariant sets

Let $T : X \rightarrow X$ be a transformation.

Definition 5.1. A set $A \subseteq X$ is said to be T -invariant if $T^{-1}A = A$.

Definition 5.2. A set $A \subseteq X$ is said to be T -invariant if

$$x \in A \iff Tx \in A.$$

Question/Problem 5.3. Are Definitions 5.1 and 5.2 equivalent?

Question/Problem 5.4. What are the invariant sets for the rotation T_θ ? Explore the question for different values of θ .

Question/Problem 5.5. What are the invariant sets for the $\times m$ map, T_m ? We saw that the Cantor set is invariant for T_3 .

Question/Problem 5.6. Can you find sets that are invariant for both T_2 and T_3 ? I can think of a couple. The set $[0, 1)$, of course, is invariant under both, as is \emptyset , vacuously. Here's another: $\{0\}$. Any others?

Question/Problem 5.7. Can you find invariant sets for toral translations? For the cat map? For symbolic systems?

5.2 More about measures

A set $A \subseteq \mathbb{R}$ is called *measurable* if for every $\varepsilon > 0$ there exists an open set \mathcal{O} such that $A \subseteq \mathcal{O}$ and $\lambda^*(\mathcal{O} - A) < \varepsilon$.

Question/Problem 5.8. Prove that every null set in \mathbb{R} is measurable.

Question/Problem 5.9. Is every closed set in \mathbb{R} measurable?

6 February 16, 2026

6.1 Some more measure theory

We stated a couple of theorems about Lebesgue measure: that it is countably additive, and that it is invariant under translations and rotations.

Question/Problem 6.1. Prove translation invariance. That is, prove that for every measurable set $A \subseteq \mathbb{R}$, and every $t \in \mathbb{R}$, we have

$$\lambda(A) = \lambda(A + t),$$

where $(A + t)$ denotes the translated set

$$A + t = \{a + t : a \in A\}.$$

Is the outer measure λ^* translation invariant on arbitrary sets $A \subseteq X$?

Recall the Vitali construction of a non-measurable set $N \subseteq [0, 1] \subseteq \mathbb{R}$. To remind you: we define an equivalence relation \sim on \mathbb{R} where $x \sim y$ if and only if $x - y \in \mathbb{Q}$, and we consider the partition

$$[0, 1] = \bigcup_{\alpha \in A} E_\alpha$$

into equivalence classes E_α , where A is an indexing set. Now the set N is defined by choosing (*choosing!*) exactly one element from each equivalence class E_α . We observe then that

$$[0, 1] \subseteq \bigsqcup_{r \in \mathbb{Q} \cap [0, 1]} (N + r) \subseteq [-1, 2].$$

Countable additivity cannot be respected by this union, therefore, N cannot be Lebesgue measurable.

Question/Problem 6.2. To ponder: what is the outer measure of N ?

6.2 Measurable dynamics

We have started introducing concepts from dynamics that involve measure. For example, a transformation $T : X \rightarrow X$ on a measure space (X, μ) is *measurable* if for every measurable $A \subseteq X$, the preimage

$$T^{-1}(A) = \{x \in X : T(x) \in A\}$$

is a measurable set. The transformation is then called *measure-preserving* if for every measurable set $A \subseteq X$ we have $\mu(T^{-1}(A)) = \mu(A)$.

Question/Problem 6.3. Verify that circle rotations and circle-expanding maps are measure-preserving. We've argued this by looking at the preimages of intervals. Is this enough? Why?

A measure-preserving transformation $T : (X, \mu) \rightarrow (X, \mu)$ is called *mixing* if for all measurable $A, B \subseteq X$, we have

$$\lim_{n \rightarrow \infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B),$$

where $T^{-n}(A)$ is the n th preimage of A , meaning the preimage of the preimage of the preimage of ...the preimage of the preimage of A under T .

Question/Problem 6.4. We saw that the circle rotation is not mixing, while the $\times 2$ map of the circle is. Are toral translations mixing? Is the cat map mixing? Is the shift on $\{0, 1\}^{\mathbb{N}}$ mixing? (What would the measure be?)

7 February 18, 2026

7.1 A fun problem about powers of 2

Take a look at the powers of 2:

1	2	4	8	16	32	64	128	256	512	1024
	2048	4096	8192	16384	32768	65536				
		131072	262144	524288	1048576	...				
				...						

Do you notice anything about their leading digits? Why don't any powers of 2 have 7 their leading digit? (Same question for 9.) Did I simply not list enough of them here, or might it be—for some number-theoretic reason—that 7 is *never* the leading digit of 2^n ?

Notice that to say that 2^n has 7 as its leading digit is to say that

$$7 \times 10^d \leq 2^n < 8 \times 10^d$$

for some integer $d \geq 0$. Equivalently,

$$d + \log_{10} 7 < n \log_{10} 2 \leq d + \log_{10} 8$$

for some integer $d \geq 0$. Equivalently,

$$n \log_{10} 2 \pmod{1} \in (\log_{10} 7, \log_{10} 8].$$

That is, 2^n has 7 as its leading digit if (and only if)

$$T_{\theta}^n(0) \in (\log_{10} 7, \log_{10} 8],$$

where $\theta = \log_{10} 2$. But isn't the T_{θ} -orbit of 0 dense? If so, this implies

Theorem. *There are infinitely many powers of 2 having 7 as their leading digit.*

Question/Problem 7.1. The proof of this theorem now only requires that $\theta = \log_{10} 2$ be irrational. Prove this.

7.2 Back to measurable dynamics

Recall that circle rotations are measure-preserving.

Question/Problem 7.2. Can you adapt Vitali's construction (of a nonmeasurable set) to the circle by using irrational circle rotations? Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Use the fact that T_θ -orbits are dense and that T_θ is measure-preserving.

We have now discussed a few different ways of saying that a transformation spreads points out throughout the phase space. For example, we have seen that irrational circle rotations have dense orbits. *Density* is some notion of "spreadedness." A set is dense if it intersects every open set.

There are also measure-theoretic notions of "spreadedness." *Mixing* is one of them. Loosely speaking, it implies that every positive-measure set's preimages intersect every other positive measure set. In fact, *mixing* says much more than just that.

An open-ended question to ponder: can you think of softer notions of "spreadedness" that can be expressed in measure-theoretic words? Maybe something similar to *are there dense orbits?*, but without topological language? One might be inclined to replace the density question

Are there orbits that enter every open set?

with the measure-theoretic analogue

Are there orbits that enter every positive measure set?

But the issue here is that orbits tend to be zero-measure sets themselves! They miss their (full measure) complements, so the answer to this new question is always *no*. What should we ask instead?

8 February 25, 2026

8.1 Recurrence

Definition 8.1. A measure-preserving transformation T on (X, μ) is called recurrent if for every $\mu(A) > 0$, almost every $x \in A$ has some $n = n(x) > 0$ such that $T^n x \in A$.

We argued that rational circle rotations are not recurrent, while irrational circle rotations seem to be.

We proved that if T is recurrent, then for every $\mu(A) > 0$, almost every member of A recurs infinitely many times to A .

Definition 8.2. A measure-preserving transformation $T : X \rightarrow X$ is called conservative if for every $\mu(A) > 0$ there exists $n > 0$ such that $\mu(A \cap T^{-n}A) > 0$.

We saw that measure-preservingness and conservativeness are equivalent, and we proved the following foundational result.

Theorem (Poincare Recurrence). *Every measure-preserving transformation on a finite measure space is recurrent.*

Definition 8.3 (Multiply recurrent transformation). A measure-preserving transformation T is called multiply recurrent if for every $\mu(A) > 0$ and $k > 0$, there exists $n > 0$ such that

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap \dots \cap T^{-(k-1)n}A) > 0.$$

Furstenberg proved the following.

Theorem (Furstenberg Multiple Recurrence). *If T is measure-preserving on a finite Lebesgue measure space (X, \mathcal{M}, μ) , then T is multiply recurrent.*

Question/Problem 8.4. Show that circle rotations are multiply recurrent.

Question/Problem 8.5. Show that circle expanding maps are multiply recurrent.

9 March 2, 2026

9.1 Natural density

Definition 9.1 (Upper and lower density; density). For $A \subseteq \mathbb{N}$ we define

$$\begin{aligned} \bar{d}(A) &= \limsup_{N \rightarrow \infty} \frac{|A \cap [0, N]|}{N} && \text{(upper density)} \\ \underline{d}(A) &= \liminf_{N \rightarrow \infty} \frac{|A \cap [0, N]|}{N}. && \text{(lower density)} \end{aligned}$$

If $\bar{d}(A) = \underline{d}(A)$, then the common value is denoted $d(A)$ and it is simply called the (*natural density*) of A .

As we noted in class, not every $A \subseteq \mathbb{N}$ has a density.

Question/Problem 9.2. Give an explicit example of a set $A \subseteq \mathbb{N}$ whose lower density is 0 and whose upper density is 1.

Question/Problem 9.3. Given $0 \leq \alpha \leq \beta \leq 1$, give an example of a set $A \subseteq \mathbb{N}$ having $\underline{d}(A) = \alpha$ and $\bar{d}(A) = \beta$.

Question/Problem 9.4. Suppose $A \subseteq \mathbb{N}$ has positive (lower/upper?) density. Then $\sum_{n \in A} 1/n = \infty$.

Question/Problem 9.5. Is the converse true? Do you think the primes have positive density?

9.2 Szemerédi's theorem

Szemerédi's Theorem. *If $\bar{d}(A) > 0$, then A contains arithmetic progressions of arbitrary length.*

Question/Problem 9.6. Is the converse true?

10 March 4, 2026

Back to dynamics!

10.1 Ergodic transformations, Ergodic theorem

Definition 10.1 (Ergodic transformation). A measure-preserving system (X, μ, T) is called *ergodic* if every invariant set is null or co-null. That is, if

$$T^{-1}(A) = A \implies \mu(A) = 0 \text{ or } \mu(X \setminus A) = 0$$

holds.

It turns out that this condition is enough to guarantee that almost all orbits equidistribute.

The Ergodic Theorem (Birkhoff, statement to be refined). *If (X, μ, T) is a finite measure ergodic system, then for every measurable set $A \subseteq X$ and almost every $x \in X$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_A(T^n x) = \frac{\mu(A)}{\mu(X)}.$$

Question/Problem 10.2. Assuming (irrational) rotations and $\times m$ maps are ergodic, and assuming the ergodic theorem, prove Benford's law for powers of 2. Prove Borel's normal numbers theorem.

10.2 Equivalent formulations

Theorem. *Let (X, μ) be a measure space and $T : X \rightarrow X$ mpt. TFAE:*

- T is recurrent and ergodic.
- If $\mu(A) > 0$ then $\mu\left(X \setminus \bigcup_{n \geq 0} T^{-n}A\right) = 0$.
- If $\mu(A) > 0$ and a.e. $x \in X$, $\exists n \geq 1, T^n x \in A$. (Discuss in connection to recurrence.)
- $\mu(A), \mu(B) > 0 \implies \exists n > 0 T^{-n}A \cap B \neq \emptyset$. (Compare to mixing).
- $\mu(A), \mu(B) > 0 \implies \exists n > 0 \mu(T^{-n}A \cap B) > 0$. (Compare to previous).

11 March 23, 25, 2026

Lots of talk of ergodicity! We've proved a theorem with different characterizations of ergodicity, and we've discussed proofs of the facts that irrational rotations and circle expanding maps are ergodic.

Question/Problem 11.1. Determine whether the baker transformation is ergodic. The cat map? The left shift on $\{0, 1\}^{\mathbb{N}}$?

Question/Problem 11.2. Let $X = \{x_1, x_2, x_3, \dots, x_N\}$, a finite set, endowed with the counting measure ν . That is, $\nu(\{x_i\}) = 1$ for all $i = 1, \dots, N$. Let $a \in \mathbb{N}$. Define the transformation $T_a : X \rightarrow X$ by

$$T_a(x_i) = x_{(i+a \bmod N)}.$$

Show that T_a is measure preserving. Under what conditions on a is T_a ergodic? Prove.

12 March 30, April 1, 2026

12.1 Ergodicity and invariant functions

We proved that for (X, μ) a finite probability space and $T : X \rightarrow X$ a measure-preserving transformation, ergodicity is equivalent to all T -invariant functions being almost everywhere constant.

Question/Problem 12.1. Prove that T is ergodic if and only if every essentially invariant set is either null or full, that is,

$$\mu(T^{-1}A \Delta A) = 0 \implies \mu(A) \in \{0, \mu(X)\}.$$

Question/Problem 12.2. Prove that T is ergodic if and only if essentially invariant function is almost everywhere constant, that is,

$$\mu(\{x : (f \circ T)(x) \neq f(x)\}) = 0 \implies \exists c, \mu(\{x : f(x) = c\}) = \mu(X).$$

Question/Problem 12.3. True or false: T is ergodic if and only if every invariant function is constant. Prove your answer. If your answer is "false," then ask yourself the follow-up question: Are there choices of (X, μ) for which it is true?

Question/Problem 12.4. Suppose (X, μ, T) is a measure-preserving system, and $f : X \rightarrow \mathbb{R}$ a measurable function. Let A_N be the averaging operator

$$A_N f = \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n.$$

In other words, $A_N f : X \rightarrow \mathbb{R}$ is the function that gives the average of

$$f(x), f(T(x)), f(T^2x), \dots, f(T^{N-1}x)$$

What can you say about the limit $\lim_{N \rightarrow \infty} A_N f$?

12.2 Functional analytic discussion

Many of the most powerful tools in ergodic theory come from functional analysis, a field where we view function spaces as vector spaces and we prove results analogous to those from linear algebra.

Some of the most important examples of these function spaces are what are called “Hilbert spaces.” Their defining property is that they have an inner product (or “dot product.”) All finite-dimensional vector spaces are Hilbert spaces, with the inner product you are familiar with. For example,

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^n x_i y_i$$

for $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$. Or

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \bar{w}_1 + z_2 \bar{w}_2 + \cdots + z_n \bar{w}_n = \sum_{i=1}^n z_i \bar{w}_i$$

for $\mathbf{z} = (z_1, \dots, z_n), \mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$, where \bar{z} denotes the complex conjugate of z .

Let $L^2([0, 1])$ denote the space of square-integrable functions $f : [0, 1] \rightarrow \mathbb{C}$:

$$L^2([0, 1]) = \left\{ f : [0, 1] \rightarrow \mathbb{C} : \int_0^1 |f|^2 dx < \infty \right\} / \sim$$

where $f \sim g$ if and only if $f = g$ almost everywhere. It is most natural (and usually harmless) to think of the elements of L^2 as functions, but they are really \sim -equivalence classes. For $f \in L^2$, define

$$\|f\| = \left(\int |f|^2 \right)^{1/2}.$$

Question/Problem 12.5. Verify that $\|\cdot\|$ defines a norm on L^2 :

- $\|f\| \geq 0$ for every $f \in L^2$
- $\|f\| = 0$ if and only if $f = 0$ in L^2 .
- For all $f \in L^2$ and $c \in \mathbb{C}$, we have $\|cf\| = |c|\|f\|$.
- For all $f, g \in L^2$, we have $\|f + g\| = \|f\| + \|g\|$.

The inner product on L^2 is similar to the inner product on \mathbb{R}^n , or \mathbb{C}^n . Given $f, g \in L^2$, define

$$(f, g) = \int f \bar{g}.$$

Question/Problem 12.6. Show that (\cdot, \cdot) satisfies the properties of an inner product:

- $f \mapsto (f, g)$ is linear for every fixed g

- $(f, g) = \overline{(g, f)}$
- $(f, f) \geq 0$ for all f
- $(f, f) = 0$ if and only if $f = 0$ (recalling that this really means $f(x) = 0$ for almost every x .)