Contents

Note

Problems come from Stein–Shakarchi [1]. Numbering is in the form

```
chapter.exercise,
```

so that Stein–Shakarchi, Exercise 2.16 refers to Exercise 16 in Chapter 2 of [1].

Due Monday, February 9, 2026

Stein–Shakarchi, Exercise 2.16

Suppose f is integrable on $\mathbb{R}^d$. If $\delta = (\delta_1, \dots, \delta_d)$ is a d-tuple of non-zero real numbers, and

$$f^\delta(x) = f(\delta x) = f(\delta_1 x_1, \dots, \delta_d x_d),$$

show that f^δ is integrable with

$$\int_{\mathbb{R}^d} f^\delta(x)\,dx = |\delta_1|^{-1} \cdots |\delta_d|^{-1} \int_{\mathbb{R}^d} f(x)\,dx. \tag{1}$$

Proof. Let us first assume that f is nonnegative.

By the scaling property of Lebesgue measure, an integrable function g on $\mathbb{R}$ satisfies

$$\int_{\mathbb{R}} g(\delta x)\,dx = \frac{1}{|\delta|} \int_{\mathbb{R}} g(x)\,dx \tag{2}$$

for every $\delta \neq 0$. In particular, the function defined by $g(\delta x)$ is also integrable.

That f^δ is measurable and nonnegative is a routine verification. Denote $y = (x_2, \dots, x_{d-1}) \in \mathbb{R}^{d-1}$ and $\delta y = (\delta_2 x_2, \dots, \delta_d x_d)$. By Fubini–Tonelli, for almost every fixed $y \in \mathbb{R}^{d-1}$, $f(\delta_1 x_1, \delta y)$ is a (nonnegative) measurable function of x_1, and

$$\int_{\mathbb{R}} f(\delta_1 x_1, \delta y)\,dx_1$$

defines a (nonnegative) measurable function of $y \in \mathbb{R}^{d-1}$. We have

$$\int_{\mathbb{R}^d} f^\delta(x)\,dx = \int_{\mathbb{R}^{d-1}} \left[\int_{\mathbb{R}} f(\delta_1 x_1, \delta y)\,dx_1 \right] dy.$$

Applying (2) to the inside integral gives

$$\int_{\mathbb{R}^d} f^\delta(x)\,dx = \frac{1}{|\delta_1|} \int_{\mathbb{R}^{d-1}} \underbrace{\left[\int_{\mathbb{R}} f(x_1, \delta y,)\,dx_1 \right]} dy.$$

We can now view the indicated expression as a function $\tilde{f}(\delta y)$ on $\mathbb{R}^{d-1}$, which is guaranteed by Tonelli to be measurable (and nonnegative), and to which we may apply the same reasoning as we have just applied to f. Repeating this d times results in (1) and, incidentally, the integrability of f^{δ}.

For the general case, we split f into f^+ and f^-. □

Stein–Shakarchi, Exercise 2.18

Let f be measurable and finite-valued on $[0,1]$, and suppose that $|f(x)-f(y)|$ is integrable on $[0,1]^2$. Show that f is integrable on $[0,1]$.

Proof. □

Stein–Shakarchi, Exercise 2.19

Suppose f is integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let

$$E_\alpha = \{x : |f(x)| > \alpha\}.$$

Prove that

$$\int_{\mathbb{R}^d} |f(x)|\,dx = \int_0^\infty m(E_\alpha)\,d\alpha.$$

Proof. We may assume without loss of generality that $f \geqslant 0$. Let

$$A = \{(x,y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leqslant y \leqslant f(x)\}$$

and recall that A is a measurable set in $\mathbb{R}^d \times \mathbb{R}$ and that

$$m(A) = \int_{\mathbb{R}^d} f(x)\,dx, \tag{3}$$

by Fubini. On the other hand,

$$\begin{aligned} m(A) &= \int_{\mathbb{R}^{d+1}} \chi_A \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \chi_A(x,y)\,dx\,dy, \end{aligned}$$

also by Fubini. But

$$\int_{\mathbb{R}^d} \chi_A(x,y)\,dx = m(E_y) + \int_{\mathbb{R}^d} \chi_{\{f(x)=y\}}(x,y)\,dx,$$

so we have shown that

$$\begin{aligned} m(A) &= \int_{\mathbb{R}} m(E_y)\,dy + \int_{\mathbb{R}} \int_{\mathbb{R}^d} \chi_{\{f(x)=y\}}(x,y)\,dx \\ &= \int_0^\infty m(E_y)\,dy. \end{aligned} \tag{4}$$

The result follows by combining (3) and (4). □

Due Monday, Feburary 16, 2026

Stein–Shakarchi, Exercise 6.1

Let X be a set and $\mathcal{M}$ a non-empty collection of subsets of X. Prove that if $\mathcal{M}$ is closed under

- complements,
- countable unions of disjoint sets, and
- finite intersections,

then $\mathcal{M}$ is a σ-algebra. Show that the conclusion is false if the third assumption is removed. *Tsk, tsk,* [1], *tsk, tsk*...

Proof. We must show that $\mathcal{M}$ is closed under arbitrary countable unions. Arbitrary countable intersections will follow.

Let $E_1, E_2, \cdots \in \mathcal{M}$. Put $E_0 = \varnothing$ and, for each $n \geqslant 1$, put

$$F_n = E_n \setminus (E_1 \cup \cdots \cup E_{n-1}).$$

Note that $F_1, F_2, \ldots$ are disjoint sets and that

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n. \tag{5}$$

We claim that $\bigcup_{n=1}^{\infty} F_n \in \mathcal{M}$. Observe that

$$\begin{aligned} F_n &= E_n \setminus (E_1 \cup \cdots \cup E_{n-1}) \\ &= E_n \cap (E_1 \cup \cdots \cup E_{n-1})^c \\ &= E_n \cap E_1^c \cap \cdots \cap E_{n-1}^c. \end{aligned}$$

Since $\mathcal{M}$ is closed under complements and finite intersections, we have $F_n \in \mathcal{M}$. And since $\mathcal{M}$ is closed under countable disjoint unions, we have $\bigcup_n F_n \in \mathcal{M}$, which implies $\bigcup_n E_n \in \mathcal{M}$ after recalling (5). This shows that $\mathcal{M}$ is closed under arbitary countable unions. σ-algebra! □

Counterexample. We show that the conclusion is false if the third assumption is removed. Take a set X with two subsets $A, B \subseteq X$, such that $A \neq B$, $A \cap B \neq \varnothing$, and $A \cup B \neq X$. Then

$$\mathcal{M} = \{\varnothing, A, A^c, B, B^c, X\}$$

satisfies the first two conditions, but it is not a σ-algebra because, for example, $A \cup B \notin \mathcal{M}$. □

Stein–Shakarchi, Exercise 6.2

Let $(X, \mathcal{M}, \mu)$ be a measure space. One can define the **completion** of this space as follows. Let $\overline{\mathcal{M}}$ be the collection of sets of the form $E \cup Z$, where $E \in \mathcal{M}$, and $Z \subseteq F$ with $F \in \mathcal{M}$ and $\mu(F) = 0$. Also, define $\overline{\mu}(E \cup Z) = \mu(E)$. Then:

(a) $\overline{\mathcal{M}}$ is the smallest σ-algebra containing $\mathcal{M}$ and all subsets of elements of $\mathcal{M}$ of measure zero.

(b) The function $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$, and this measure is complete.

Proof of (a). Clearly $\overline{\mathcal{M}}$ is a nonempty collection. Suppose $E_1 \cup Z_1, E_2 \cup Z_2, \cdots \in \overline{\mathcal{M}}$, with $Z_n \subseteq F_n$ and $\mu(F_n) = 0$. Then

$$\bigcup_n (E_n \cup Z_n) = \underbrace{\left(\bigcup_n E_n\right)}_{\in \mathcal{M}} \cup \underbrace{\left(\bigcup_n Z_n\right)}_{\subseteq (\bigcup_n F_n)}.$$

Since $\mu\left(\bigcup_n F_n\right) = 0$, this shows that $\overline{\mathcal{M}}$ is closed under countable unions. To see that it is closed under complements, let $E \cup Z \in \overline{\mathcal{M}}$ with $Z \subseteq F$ and $\mu(F) = 0$. Then

$$\begin{aligned}(E \cup Z)^c &= E^c \cap Z^c \\ &= (E^c \cap Z^c \cap F^c) \cup (E^c \cap Z^c \cap F) \\ &= \underbrace{(E^c \cap F^c)}_{\in \mathcal{M}} \cup \underbrace{(E^c \cap Z^c \cap F)}_{\subseteq F} \\ &\in \overline{\mathcal{M}}\end{aligned}$$

and this shows that $\overline{\mathcal{M}}$ is a σ-algebra.

Now, suppose $\mathcal{M}'$ is a σ-algebra containing $\mathcal{M}$ and all subsets of its null sets. We must show $\overline{\mathcal{M}} \subseteq \mathcal{M}'$. This is immediate. Every member of $\overline{\mathcal{M}}$ is a union of two members of $\mathcal{M}'$. □

Proof of (b). We must show that $\overline{\mu} : \overline{\mathcal{M}} \to [0, \infty]$ is well-defined and countably additive.

Suppose $E \cup Z = E' \cup Z'$, with $Z \subseteq F$ and $Z' \subseteq F'$, $F, F' \in \mathcal{M}$ null sets. Since $E' \subseteq E \cup Z \subseteq E \cup F$, we have

$$\mu(E') \leqslant \mu(E \cup F) \leqslant \mu(E) + \mu(F) = \mu(E)$$

The opposite inequality is similar, so $\mu(E') = \mu(E)$, hence $\overline{\mu}$ is well-defined.

Suppose $(E_n \cup Z_n)\,(n \geqslant 1)$ are disjoint members of $\overline{\mathcal{M}}$, with $Z_n \subseteq F_n$ for $\mu(F_n) = 0$. Then

$$\begin{aligned}\overline{\mu}\left(\bigcup_{n=1}^{\infty}(E_n \cup Z_n)\right) &= \overline{\mu}\left(\underbrace{\left(\bigcup_{n=1}^{\infty} E_n\right)}_{\in \mathcal{M}} \cup \underbrace{\left(\bigcup_{n=1}^{\infty} Z_n\right)}_{\subseteq \bigcup_n F_n}\right) \\ &= \mu\left(\bigcup_{n=1}^{\infty} E_n\right) \\ &= \sum_{n=1}^{\infty} \mu(E_n) \\ &= \sum_{n=1}^{\infty} \overline{\mu}(E_n \cup Z_n),\end{aligned}$$

and done! □

Stein–Shakarchi, Exercise 6.3

Consider the exterior Lebesgue measure m_* introduced in Chapter 1. Prove that a set $E \in \mathbb{R}^d$ is Carathéodory measurable if and only if E is Lebesgue measurable in the sense of Chapter 1.

One direction. Suppose E is Lebesgue measurable and let $A \subseteq \mathbb{R}^d$. Let $\varepsilon > 0$, and $\mathcal{O}$ an open set containing E such that $m_*(\mathcal{O} - E) < \varepsilon$. Then

$$\begin{aligned} m_*(A) \leqslant m_*(A \cap E) + m_*(A \cap E^c) &\leqslant m_*(A \cap \mathcal{O}) + m_*(A \cap E^c) \\ &\leqslant m_*(A \cap \mathcal{O}) + m_*(A \cap E^c \cap \mathcal{O}) + m_*(A \cap E^c \cap \mathcal{O}^c) \\ &= m_*(A \cap \mathcal{O}) + \underbrace{m_*(A \cap (\mathcal{O} - E))}_{<\varepsilon} + m_*(A \cap \mathcal{O}^c) \\ &= m_*(A) + \varepsilon, \end{aligned}$$

by Caratheodory measurability of open sets. Since $\varepsilon > 0$ is arbitrary, this shows that E is Carathéodory measurable. □

Other direction. Suppose $E \subseteq \mathbb{R}^d$ is Caratheodory measurable. Let $N \in \mathbb{N}$ and $\varepsilon > 0$. Let $\mathcal{O}$ be an open set containing $E_N := E \cap [-N, N]^d$ such that

$$m_*(\mathcal{O}) \leqslant m_*(E_N) + \varepsilon.$$

But we also have

$$\begin{aligned} m_*(\mathcal{O}) &= m_*(\mathcal{O} \cap E_N) + m_*(\mathcal{O} \cap E_N^c) \\ &= m_*(E_N) + m_*(\mathcal{O} - E_N). \end{aligned}$$

Combining, we get $m_*(\mathcal{O} - E_N) \leqslant \varepsilon$. This shows that E_N is Lebesgue measurable. Therefore, $E = \bigcup_{N \in \mathbb{N}} E_N$ is Lebesgue measurable. □

Due Monday, Feburary 23, 2026

Stein–Shakarchi, Exercise 6.13

Let m_j be the Lebesgue measure for the space $\mathbb{R}^{d_j}$, $j = 1, 2$. Consider the product $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ ($d = d_1 + d_2$), with m the Lebesgue measure on $\mathbb{R}^d$. Show that m is the completion (in the sense of Stein–Shakarchi, Exercise 6.2) of the product measure $m_1 \times m_2$.

Proof. First, we show that $\mathcal{L}_1 \otimes \mathcal{L}_2 \subseteq \mathcal{L}$ and

$$m \mid_{\mathcal{L}_1 \otimes \mathcal{L}_2} = m_1 \times m_2,$$

where $\mathcal{L} := \mathcal{L}_{\mathbb{R}^d}$, $\mathcal{L}_1 := \mathcal{L}_{\mathbb{R}^{d_1}}$, and $\mathcal{L}_2 := \mathcal{L}_{\mathbb{R}^{d_2}}$ (the Lebesgue measurable sets). To see this, recall that $\mathcal{L}_1 \otimes \mathcal{L}_2$ is the smallest σ-algebra containing the measurable rectangles $A \times B$ ($A \in \mathcal{L}_1, B \in \mathcal{L}_2$). By [1, Chapter 2, Proposition 3.6], we have $A \times B \in \mathcal{L}$ and

$$m(A \times B) = m_1(A) m_2(B).$$

This extends uniquely to a premeasure on the algebra $\mathcal{A}$ of finite unions of measurable rectangles. Since $m_1 \times m_2$ is σ-finite, it is the unique extension to $\mathcal{L}_1 \otimes \mathcal{L}_2$ of the pre-measure on $\mathcal{A}$ ([1, Chapter 6, Theorem 1.5]). It follows that m and $m_1 \times m_2$ must coincide on $\mathcal{L}_1 \otimes \mathcal{L}_2$.

We now have that m extends $m_1 \times m_2$ from $\mathcal{L}_1 \otimes \mathcal{L}_2$ to $\mathcal{L}$. Let $\mathcal{L}'$ be the completion of $\mathcal{L}_1 \otimes \mathcal{L}_2$ in the sense of Stein–Shakarchi, Exercise 6.2. We must show that $\mathcal{L}' = \mathcal{L}$. Since $\mathcal{L}$ is complete, it in particular contains all subsets of the null sets of $\mathcal{L}_1 \otimes \mathcal{L}_2$, as well as $\mathcal{L}_1 \otimes \mathcal{L}_2$ itself, as we have finished showing. Now, by Stein–Shakarchi, Exercise 6.2, it follows that $\mathcal{L}' \subseteq \mathcal{L}$. On the other hand, $\mathcal{L}$ is the completion of $\mathcal{B}$, the Borel sets, and $\mathcal{B} \subseteq \mathcal{L}_1 \otimes \mathcal{L}_2$. Since the latter is complete, we must have $\mathcal{L} \subseteq \mathcal{L}'$ □

Stein–Shakarchi, Exercise 6.14

Suppose $(X_j, \mathcal{M}_j, \mu_j)$, $1 \leqslant j \leqslant k$, is a finite collection of measure spaces. Show that parallel with the case $k = 2$ considered in Section 3 one can construct a product measure $\mu_1 \times \mu_2 \times \cdots \times \mu_k$ on $X = X_1 \times X_2 \times \cdots \times X_k$. In fact, for any set $E \subseteq X$ such that $E = E_1 \times E_2 \times \cdots \times E_k$, with $E_j \in \mathcal{M}_j$ for all j, define $\mu_0(E) = \prod_{j=1}^{k} \mu_j(E_j)$. Verify that μ_0 extends to a premeasure on the algebra $\mathcal{A}$ of finite disjoint unions of such sets, and then apply Theorem 1.5.

Lemma 1. *If* $F = A_1 \times \cdots \times A_k$ *with* $A_j \in \mathcal{M}_j$ *and*

$$F = \bigsqcup_{n=1}^{\infty} E_n,$$

where $E_n = A_1^{(n)} \times \cdots \times A_k^{(n)}$ *with* $A_j^{(n)} \in \mathcal{M}_j$, *then*

$$\mu_0(F) = \sum_{n=1}^{\infty} \mu_0(E_n). \tag{6}$$

Proof of Stein–Shakarchi, Exercise 6.14 modulo Lemma 1. We extend μ_0 to $\mathcal{A}$ by putting

$$\mu_0(E) = \sum_{n=1}^{N} \mu_0(E_n)$$

whenever $E = \bigsqcup_{n=1}^{N} E_n$ with $E_n = E_1^{(n)} \times \cdots \times E_k^{(n)}$, and $E_j^{(n)} \in \mathcal{M}_j$.

We should first verify that this is well-defined. Suppose $E \in \mathcal{A}$ can also be expressed $E = \bigsqcup_{m=1}^{M} F_m$ with $F_m = F_1^{(m)} \times \cdots \times F_k^{(m)}$, and $F_j^{(m)} \in \mathcal{M}_j$. We must show that

$$\sum_{m=1}^{M} \mu_0(F_m) = \sum_{n=1}^{N} \mu_0(E_n) \tag{7}$$

But for each m, we have

$$F_m = \bigsqcup_{n=1}^{N} E_n \cap F_m,$$

and for each n, the set $E_n \cap F_m$ is a measurable rectangle. Therefore, by Lemma 1,

$$\mu_0(F_m) = \sum_{n=1}^{N} \mu_0(E_n \cap F_m)$$

hence

$$\sum_{m=1}^{M} \mu_0(F_m) = \sum_{m=1}^{M} \sum_{n=1}^{N} \mu_0(E_n \cap F_m).$$

A symmetric argument shows

$$\sum_{n=1}^{N} \mu_0(E_n) = \sum_{n=1}^{N} \sum_{m=1}^{M} \mu_0(E_n \cap F_m),$$

and (7) follows. The extension of μ_0 to $\mathcal{A}$ is well-defined.

That $\mu_0(\varnothing) = 0$ is inherited. In order to see that μ_0 is a premeasure on $\mathcal{A}$ we must show that if $A_1, A_2, \dots, \in \mathcal{A}$ are disjoint sets such that $\bigcup_n A_n \in \mathcal{A}$, then

$$\mu_0\left(\bigcup_n A_n\right) = \sum_n \mu_0(A_n).$$

Since $\bigcup_n A_n \in \mathcal{A}$, it is expressible as a union of finitely many disjoint rectangles,

$$\bigcup_n A_n = \bigsqcup_{m=1}^{M} B_m,$$

and

$$\mu_0\Big(\bigcup_n A_n\Big) = \sum_{m=1}^{M} \mu_0(B_m).$$

The problem thus reduces to showing

$$\sum_n \mu_0(A_n) = \sum_{m=1}^{M} \mu_0(B_m).$$

Since Lemma 1 holds for countable disjoint unions, the arguments from the previous paragraph can be repeated here. □

Proof of Lemma 1. Observe that for $x = (x_1, x_2, \dots, x_k) \in X$, we have

$$\chi_{A_1\times\cdots\times A_k}(x) = \sum_{n=1}^{\infty} \chi_{A_1^{(n)}\times\cdots\times A_k^{(n)}}(x)$$

$$\updownarrow$$

$$\prod_{j=1}^{k} \chi_{A_j}(x_j) = \sum_{n=1}^{\infty} \prod_{j=1}^{k} \chi_{A_j^{(n)}}(x_j).$$

Fixing $(x_2, \dots, x_k) \in X_2 \times X_3 \times \cdots \times X_k$, we integrate both sides with respect to μ_1. The monotone convergence theorem gives

$$\int \chi_{A_1}(x_1) \prod_{j=2}^{k} \chi_{A_j}(x_j)\, d\mu_1(x_1) = \sum_{n=1}^{\infty} \int \chi_{A_1^{(n)}}(x_1) \prod_{j=2}^{k} \chi_{A_j^{(n)}}(x_j)\, d\mu(x_1),$$

and, evaluating integrals,

$$\mu_1(A_1) \prod_{j=2}^{k} \chi_{A_j}(x_j) = \sum_{n=1}^{\infty} \mu_1(A_1^{(n)}) \prod_{j=2}^{k} \chi_{A_j^{(n)}}(x_j).$$

Now fix $(x_3, \dots, x_k)$ and integrate both sides with respect to μ_2. This leads via the monotone convergence theorem to

$$\mu_1(A_1)\mu_2(A_2) \prod_{j=3}^{k} \chi_{A_j}(x_j) = \sum_{n=1}^{\infty} \mu_1(A_1^{(n)})\mu_2(A_2^{(n)}) \prod_{j=3}^{k} \chi_{A_j^{(n)}}(x_j).$$

Continuing in this way, we arrive after k steps at

$$\prod_{j=1}^{k} \mu_j(A_j) = \sum_{n=1}^{\infty} \prod_{j=1}^{k} \mu_j(A_j^{(n)}),$$

which is equivalent to (6). This proves Lemma 1. □

Stein–Shakarchi, Exercise 6.16

Due Friday, March 6, 2026

Stein–Shakarchi, Exercise 3.4

Prove that if f is integrable on $\mathbb{R}^d$, and f is not identically zero, then

$$f^*(x) \geqslant \frac{c}{|x|^d}, \qquad \text{for some } c > 0 \text{ and all } |x| \geqslant 1. \tag{8}$$

Conclude that f^* is not integrable on $\mathbb{R}^d$. Then, show that the weak type estimate

$$m(\{f^* > \alpha\}) \leqslant c/\alpha$$

for all $\alpha > 0$ whenever $\int |f| = 1$, is best possible in the following sense: if f is supported in the unit ball with $\int |f| = 1$, then

$$m(\{f^* > \alpha\}) \geqslant c'/\alpha$$

for some $c' > 0$ and all sufficiently small α.

Hint. For the first part, use the fact that $\int_B |f| > 0$ for some ball B. □

Proof. That f is integrable and not "identically" zero means that $\|f\|_1 \neq 0$. In particular, there exists a ball B for which $\int_B |f| > 0$, as the hint points out. In fact, by increasing the radius of B, we may assume that $r := \max\{|y| : y \in B\} > 1$.

Now, whenever $|x| \geqslant 1$, the ball $B(0, r|x|)$ contains both x and B. In particular, we have for $|x| \geqslant 1$ that

$$\begin{aligned} f^*(x) &\geqslant \frac{1}{m(B(0,r|x|))} \int_{B(0,r|x|)} |f| \\ &\geqslant \frac{1}{m(B(0,r|x|))} \int_B |f| \\ &= \frac{\int_B |f|}{m(B(0,r))} \cdot \frac{1}{|x|^d}. \end{aligned}$$

This proves (8) with $c = \frac{\int_B |f|}{m(B(0,r))} > 0$, which in turn implies that f^* is not integrable.

In fact, we may show that the weak-type Hardy-Littlewood inequality is best possible. The previous paragraph shows that for all $|x| \geqslant 1$ we have

$$f^*(x) \geqslant \frac{c}{|x|^d}.$$

Therefore, if $\alpha < c$, then $\{f^* > \alpha\}$ contains the annulus $\{1 \leqslant |x| < (c/\alpha)^{1/d}\}$. That annulus in turn has measure

$$m\{1 \leqslant |x| < (c/\alpha)^{1/d}\} = \Big(\frac{c}{\alpha} - 1\Big) m(B(0,1)).$$

Hence,

$$m\{f^* > \alpha\} \geqslant \Big(\frac{c}{\alpha} - 1\Big) m(B(0,1))$$

It is easily shown that

$$\Big(\frac{c}{\alpha} - 1\Big) m(B(0,1)) \geqslant \frac{\frac{1}{2} c m(B(0,1))}{\alpha}$$

for all sufficiently small $\alpha > 0$. Setting $c' = \frac{1}{2} c m(B(0,1))$, we have that

$$m\{f^* > \alpha\} \geqslant \frac{c'}{\alpha}$$

for all sufficiently small α. □

Stein–Shakarchi, Exercise 3.7

Prove that if a measurable subset E of $[0,1]$ satisfies

$$m(E \cap I) \geqslant \alpha m(I)$$

for some $\alpha > 0$ and all intervals I in $[0,1]$, then E has measure 1. See also [1, Exercise 1.28].

Proof. If $\alpha \geqslant 1$ then the result follows by taking $I = [0,1]$. So let us assume that $0 < \alpha < 1$.

Suppose x is a density point of $E^c = [0,1] \setminus E$. Then there exists an interval I containing x which is small enough that

$$\frac{m(E^c \cap I)}{m(I)} > 1 - \alpha.$$

This implies that

$$\frac{m(\mathrm{E} \cap \mathrm{I})}{m(\mathrm{I})} < \alpha,$$

which contradicts the assumption on E. Therefore, E^c must not have any density points. But the Lebesgue density theorem guarantees that almost every point of E^c is a density point. Evidently, E^c has zero measure, hence $m(\mathrm{E}) = 1$. □

Stein–Shakarchi, Exercise 3.8

Suppose A is a Lebesgue measurable set in $\mathbb{R}$ with $m(\mathrm{A}) > 0$. Does there exist a sequence $\{s_n\}_{n=1}^{\infty}$ such that the complement of

$$\bigcup_{n=1}^{\infty}(\mathrm{A} + s_n)$$

in $\mathbb{R}$ has measure zero?

Hint. For every $\varepsilon > 0$, find an interval I_ε of length ℓ_ε such that $m(\mathrm{A} \cap \mathrm{I}_\varepsilon) \geqslant (1-\varepsilon)m(\mathrm{I}_\varepsilon)$. Consider $\bigcup_{k=-\infty}^{\infty}(\mathrm{A} + t_k)$, with $t_k = k\ell_\varepsilon$. Then vary ε. □

Proof. The Lebesgue density theorem guarantees the existence of some point $x \in \mathrm{A}$ such that

$$\lim_{\substack{|\mathrm{I}| \to 0 \\ x \in \mathrm{I}}} \frac{m(\mathrm{A} \cap \mathrm{I})}{m(\mathrm{I})} = 1.$$

In particular, we may find a sequence of intervals $\mathrm{I}_0, \mathrm{I}_1, \mathrm{I}_2, \ldots$ containing x such that $|\mathrm{I}_n| \to 0$ and such that

$$\frac{m(\mathrm{A} \cap \mathrm{I}_n)}{m(\mathrm{I}_n)} \geqslant \frac{1}{2}$$

for each $n \geqslant 0$. For each n, let

$$\mathcal{A}_n = \bigcup_{k \in \mathbb{Z}} (\mathrm{A} + k|\mathrm{I}_n|).$$

Notice that for any interval I with $|\mathrm{I}| \geqslant |\mathrm{I}_n|$, we have

$$\frac{m(\mathcal{A}_n \cap \mathrm{I})}{m(\mathrm{I})} \geqslant \frac{1}{3}. \tag{9}$$

Finally, let

$$\mathcal{A} = \bigcup_{n \geqslant 0} \mathcal{A}_n,$$

and notice that $\mathcal{A}$ is the union of countably many translates of A. Since $|I_n| \to 0$, it is guaranteed by (9) that for any nontrivial interval I,

$$\frac{m(\mathcal{A} \cap I)}{m(I)} \geqslant \frac{1}{3}.$$

In particular, this holds for all $I \subseteq [0, 1]$, and so by applying Stein–Shakarchi, Exercise 3.7 we find that $m(\mathcal{A} \cap [0, 1]) = 1$. □

Stein–Shakarchi, Exercise 3.9

Let F be a closed subset in $\mathbb{R}$, and $\delta(x)$ the distance from x to F. Prove

$$\delta(x + y) = o(|y|) \quad (|y| \to 0) \tag{10}$$

for almost every $x \in F$.

Proof. Suppose (10) does not hold for x. Then there exists $\varepsilon > 0$ such that

$$\frac{\delta(x + y)}{|y|} \geqslant 2\varepsilon$$

for arbitrarily small y. For all such y, we have

$$m(F \cap (x - y, x + y)) \leqslant 2|y| - 2\varepsilon|y|,$$

hence

$$\liminf_{|y| \to 0} \frac{m(F \cap (x - y, x + y))}{2|y|} \leqslant 1 - \varepsilon$$

and x is not a density point of F. Since almost every point of F is a density point of F, it must be that (10) holds for almost every $x \in F$. □

Due Monday, March 30, 2026

Stein–Shakarchi, Exercise 6.10

Suppose ν, ν_1, ν_2 are signed measures on $(X, \mathcal{M})$ and μ a (positive) measure on M. Prove:

(a) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $\nu_1 + \nu_2 \perp \mu$

(b) If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$

(c) $\nu_1 \perp \nu_2$ implies $|\nu_1| \perp |\nu_2|$

(d) If $\nu \perp \mu$ and $\nu \ll \mu$ then $\nu = 0$.

Proof (a). □

Stein–Shakarchi, Exercise 6.11

Suppose that F is an increasing normalized function on $\mathbb{R}$, and let $F = F_A + F_C + F_J$ be the decomposition of F in Exercise 24 in Chapter 3; here F_A is absolutely continuous, F_C is continuous with $F_C' = 0$ a.e., and F_J is a pure jump function. Let $\mu = \mu_A + \mu_C + \mu_J$ with μ, μ_A, μ_C, and μ_J the Borel measures associated to F, F_A, F_C, and F_J respectively. Verify that:

(i) μ_A is absolutely continuous with respect to Lebesgue measure and $\mu_A(E) = \int_E F'(x)\,dx$ for every Lebesgue measurable set E.

(ii) As a result, if F is absolutely continuous, then $\int f\,d\mu = \int f\,dF = \int f(x)F'(x)\,dx$ whenever f and fF' are integrable.

(iii) $\mu_C + \mu_J$ and Lebesgue measure are mutually singular.

Due Monday, April 20, 2026

Stein–Shakarchi, Exercise 4.1

Stein–Shakarchi, Exercise 4.4

Stein–Shakarchi, Exercise 4.5

Stein–Shakarchi, Exercise 4.6

Stein–Shakarchi, Exercise 4.10

Stein–Shakarchi, Exercise 4.11

References

[1] E. Stein and R. Shakarchi *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*. Princeton Lectures in Analysis